Polyhedral Realizations of Crystal Bases for Quantized Kac-Moody Algebras

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1 Introduction

Since pioneering works of G.Lusztig and M.Kashiwara on special bases for quantum groups, a lot of work has been done on the combinatorial structure of these bases. Although Lusztig's canonical bases and Kashiwara's global crystal bases were shown by Lusztig to coincide whenever both are defined, their constructions are quite different and lead to different combinatorial parametrizations. In this paper we will only discuss the basis in the q-deformation $U_q^-(\mathfrak{g})$ of the universal enveloping algebra of the nilpotent part of a Kac-Moody Lie algebra $\mathfrak{g}$; understanding this basis is an essential first step towards understanding the bases in all the integrable highest weight modules of $\mathfrak{g}$. Lusztig's approach (see [17] and references there) works especially well when $\mathfrak{g}$ is a semisimple Lie algebra of simply-laced type. In this case, every reduced expression for the maximal element of the Weyl group gives rise to a bijective parametrization of the canonical basis in $U_q^-(\mathfrak{g})$ by the semigroup $\mathbf{Z}_{\geq 0}^N$ of all N-tuples of non-negative integers, where N is the number of positive roots of $\mathfrak{g}$. These parametrizations were studied in Lusztig's papers and also in [1].

Kashiwara's construction [6] of the global crystal basis in $U_q^-(\mathfrak{g})$ is more elementary and works for an arbitrary Kac-Moody algebra. The price for this is that the parametrizing sets for the basis in Kashiwara's approach are more complicated than just $\mathbf{Z}_{\geq 0}^N$. In the literature, one can find several kinds of combinatorial expressions for crystal bases. As shown in [12],[18], crystal bases of finite-dimensional simple modules for classical Lie algebras can be parametrized by Young tableaux and their analogues. For affine Lie algebras, crystal bases of integrable highest weight modules can be expressed as infinite sequences of perfect crystals ([10],[11]) or extended Young diagrams ([5]). More generaly, in [15], [16] for any symmetrizable Kac-Moody algebra, crystal bases are realized in terms of certain polygonal paths. Although this presentation is elegant, it is not very convenient for actual computations with the basis.

In this paper we study bijective parametrizations of the crystal basis for $U_q^-(\mathfrak{g})$ by integer sequences satisfying certain linear inequalities. In more geometric terms, the basis vectors should be parametrized by lattice points in some polyhedral convex cone; this is what we mean by "polyhedral realiza-

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tions" in the title of the paper. If $\mathfrak{g}$ is semisimple, then, similarly to Lusztig's parametrizations, a polyhedral realization is naturally associated with every reduced expression for the maximal element of the Weyl group. This can be done using Kashiwara's theory of tensor products of crystals, or, equivalently, using the "string parametrizations" studied in [2], [3]. In fact, such a realization makes sense for arbitrary Kac-Moody algebras, where reduced expressions for the maximal element of the Weyl group are replaced by certain infinite sequences of indices.

In this paper we deal with the following problem: describe explicitly complete systems of linear inequalities that define all polyhedral realizations of the crystal basis in $U_q^-(\mathfrak{g})$. Such a description was recently found by P. Littelmann (private communication) for all semisimple Lie algebras, using a case-by-case analysis. Littelmann only treats some specific choice of a reduced expression (for the type A_n, the corresponding result was already obtained in [2]). We would like to find a unified description of all polyhedral realizations for an arbitrary Kac-Moody algebra. For Kac-Moody algebras of rank 2, such a description was found by M.Kashiwara in [8], Proposition 2.2.3 (this description is sharpened in Theorem 4.1 below). The main result of the present paper (Theorem 3.1 below) is a generalization of this description to Kac-Moody algebras of an arbitrary rank. The answer is given in terms of certain piecewise-linear transformations of $\mathbf{Z}^\infty$. Unfortunately, our main result is only proved under certain technical assumptions. These assumptions are checked to be valid in many cases including the ordinary and affine Lie algebras of type A. It is even conceivable that they are always satisfied (see the discussion in Section 3 below).

The paper is organized as follows. In section 2, we review crystals and their basic properties. We also introduce our main object of study: the crystal $B(\infty)$ corresponding to $U_q^-(\mathfrak{g})$. We then describe the Kashiwara embedding of $B(\infty)$ into the lattice $\mathbf{Z}^\infty$. Our main theorem is formulated and proved in Section 3: this is a description of the image of the Kashiwara embedding. In the rest of the paper (sections 4, 5, and 6), we apply the theorem to the cases when $\mathfrak{g}$ is of rank 2, of type A_n, and of type $A_{n-1}^{(1)}$, respectively.

In preparing this paper, we received the preprint "Crystal bases and Young Tableaux" by G.Cliff. Thit paper describes the image of the Kashiwara embedding for the types A, B, C, D and some special reduced expressions. The method used in the preprint is different from ours. It seems that the method cannot be applied to affine algebras or more general Kac-Moody algebras, or even to other reduced expressions for classical Lie algebras. In a forthcoming paper, polyhedral realizations will be described not only for $B(\infty)$ but also for the irreducible $\mathfrak{g}$-modules.

2 Preliminaries on Crystals

2.1 Definition of $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbf{Q}$ with a Cartan subalgebra $\mathfrak{t}$, a weight lattice $P \subset \mathfrak{t}^*$, the set of simple roots $\{\alpha_i : i \in I\} \subset \mathfrak{t}^*$, and the set of coroots $\{h_i : i \in I\} \subset \mathfrak{t}$, where I is a finite index set (see [13] for the background on Kac-Moody algebras). Let $\langle h, \lambda \rangle$ be the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$, and (α, β) be an inner product on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbf{Z}_{\geq 0}$ and $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $\lambda \in \mathfrak{t}^*$. Let $P^* = \{h \in \mathfrak{t} : \langle h, P \rangle \subset \mathbf{Z}\}$.

As in [7], we define the quantized enveloping algebra $U_q(\mathfrak{g})$ to be an associative $\mathbf{Q}(q)$-algebra generated by the e_i, f_i $(i \in I)$, and q^h $(h \in P^*)$ satisfying the following relations:

$$q^0 = 1, \quad \text{and} \quad q^h q^{h'} = q^{h+h'}, \tag{2.1}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \qquad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i, \tag{2.2}$$

$$e_i f_j - f_j e_i = \delta_{i,j}(t_i - t_i^{-1})/(q_i - q_i^{-1}), \tag{2.3}$$

$$\sum_{k=1}^{1 - \langle h_i, \alpha_j \rangle} (-1)^k x_i^{(k)} x_j x_i^{(1 - \langle h_i, \alpha_j \rangle - k)} = 0, \qquad (i \neq j) \tag{2.4}$$

where the symbol x_i in (2.4) stands for e_i or f_i, and we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, $t_i = q_i^{h_i}$, $[l]_i = (q_i^l - q_i^{-l})/(q_i - q_i^{-1})$, $[k]_i! = \prod_{l=1}^{k}[l]_i$, and $x_i^{(k)} = x_i^k/[k]_i!$.

It is well-known [4] that $U_q(\mathfrak{g})$ has a Hopf algebra structure with the comultiplication Δ given by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i,$$

for any $i \in I$ and $h \in P^*$. By this comultiplication, the tensor product of $U_q(\mathfrak{g})$-modules has a $U_q(\mathfrak{g})$-module structure.

2.2 Definition of crystals

The following definition of a crystal is due to M.Kashiwara [8, 9]; it is motivated by abstracting some combinatorial properties of crystal bases. In what follows we fix a finite index set I and a weight lattice P as above.

Definition 2.1 *A crystal B is a set endowed with the following maps:*

$$wt : B \longrightarrow P, \tag{2.5}$$

$$\varepsilon_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\} \quad \text{for} \quad i \in I, \tag{2.6}$$

$$\tilde{e}_i : B \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I. \tag{2.7}$$

Here 0 is an ideal element which is not included in B. These maps must satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle, \tag{2.8}$$

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i \ \ if \ \tilde{e}_i b \in B, \tag{2.9}$$

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i \ \ if \ \tilde{f}_i b \in B, \tag{2.10}$$

$$\tilde{e}_i b_2 = b_1 \ \ if \ and \ only \ if \ \tilde{f}_i b_1 = b_2, \tag{2.11}$$

$$if \ \varepsilon_i(b) = -\infty, \ \ then \ \tilde{e}_i b = \tilde{f}_i b = 0. \tag{2.12}$$

The above axioms allow us to make a crystal B into a colored oriented graph with the set of colors I. This means that each edge of the graph is labeled with some $i \in I$; we write $b_1 \overset{i}{\longrightarrow} b_2$ for an oriented edge from b_1 to b_2 labeled with i.

Definition 2.2 *The crystal graph of a crystal B is a colored oriented graph given by the rule : $b_1 \overset{i}{\longrightarrow} b_2$ if and only if $b_2 = \tilde{f}_i b_1$ ($b_1, b_2 \in B$).*

Definition 2.3 *(i) Let B_1 and B_2 be crystals. A morphism of crystals $\psi : B_1 \longrightarrow B_2$ is a map $\psi : B_1 \longrightarrow B_2 \sqcup \{0\}$ satisfying the following conditions:*

$$wt(\psi(b)) = wt(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \tag{2.13}$$
$$if \ b \in B_1 \ and \ \psi(b) \in B_2,$$

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b) \ if \ b \in B_1 \ satisfies \ \psi(b) \neq 0 \ and \ \psi(\tilde{e}_i b) \neq 0, \tag{2.14}$$

$$\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \ if \ b \in B_1 \ satisfies \ \psi(b) \neq 0 \ and \ \psi(\tilde{f}_i b) \neq 0. \tag{2.15}$$

(ii) A morphism of crystals $\psi : B_1 \longrightarrow B_2$ is called strict if the map $\psi : B_1 \longrightarrow B_2 \sqcup \{0\}$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ An injective strict morphism is called an embedding of crystals.

For crystals B_1 and B_2, we define their tensor product $B_1 \otimes B_2$ as follows:

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 : b_1 \in B_1, \ b_2 \in B_2\}, \tag{2.16}$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2), \tag{2.17}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle), \tag{2.18}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2) \rangle), \tag{2.19}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & if \ \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & if \ \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{2.20}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & if \ \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & if \ \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{2.21}$$

Here $b_1 \otimes b_2$ is just another notation for an ordered pair (b_1, b_2), and we set $b_1 \otimes 0 = 0 \otimes b_2 = 0$. Let $\mathcal{C}(I, P)$ be the category of crystals with the index set I

and the weight lattice P. Then $\otimes$ is a functor from $\mathcal{C}(I, P) \times \mathcal{C}(I, P)$ to $\mathcal{C}(I, P)$ that makes $\mathcal{C}(I, P)$ a tensor category [14]. In particular, the tensor product of crystals is associative: the crystals $(B_1 \otimes B_2) \otimes B_3$ and $B_1 \otimes (B_2 \otimes B_3)$ are isomorphic via $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$.

We conclude this subsection with an example of a crystal that will be needed later.

Example 2.4 *For $i \in I$, the crystal $B_i := \{(x)_i \ : \ x \in \mathbf{Z}\}$ is defined by*

$$
\begin{aligned}
wt((x)_i) &= x\alpha_i, & \varepsilon_i((x)_i) &= -x, & \varphi_i((x)_i) &= x, \\
\varepsilon_j((x)_i) &= -\infty, & \varphi_j((x)_j) &= -\infty \quad \text{for} \quad j \neq i, \\
\tilde{e}_i(x)_i &= (x+1)_i, & \tilde{f}_i(x)_i &= (x-1)_i, \\
\tilde{e}_j(x)_i &= \tilde{f}_j(x)_i = 0 \quad \text{for} \quad j \neq i.
\end{aligned}
$$

2.3 Crystal base of $U_q^-(\mathfrak{g})$ and the crystal $B(\infty)$

In this subsection we introduce the crystal $B(\infty)$, our main object of study. All the results below are due to M.Kashiwara [7]. Let $U_q^-(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $\{f_i\}_{i \in I}$. By Lemma 3.4.1 in [7], for any $u \in U_q^-(\mathfrak{g})$ and $i \in I$, there exist unique $u', u'' \in U_q^-(\mathfrak{g})$ such that

$$
e_i u - u e_i = \frac{t_i u'' - t_i^{-1} u'}{q_i - q_i^{-1}}. \tag{2.22}
$$

We define the endomorphisms e_i' and e_i'' of $U_q^-(\mathfrak{g})$ by setting $e_i'(u) = u'$ and $e_i''(u) = u''$. For any $i \in I$, we have the direct sum decomposition

$$
U_q^-(\mathfrak{g}) = \bigoplus_{k \geq 0} f_i^{(k)} \operatorname{Ker} e_i' \quad . \tag{2.23}
$$

Using this, we can define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ of $U_q^-(\mathfrak{g})$ by

$$
\tilde{e}_i(f_i^{(k)} u) = f_i^{(k-1)} u, \text{ and } \tilde{f}_i(f_i^{(k)} u) = f_i^{(k+1)} u \quad \text{for } u \in \operatorname{Ker} e_i'. \tag{2.24}
$$

Let $A \subset \mathbf{Q}(q)$ be the subring of rational functions that are regular at $q = 0$. Let $L(\infty)$ be the left A-submodule of $U_q^-(\mathfrak{g})$ generated by all the elements $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \cdot 1$ with $l \geq 0$ and $i_j \in I$. Then $L(\infty)/qL(\infty)$ is a $\mathbf{Q}$-vector space. We define a subset $B(\infty) \subset L(\infty)/qL(\infty)$ to be the set of all non-zero elements of the form $\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} \cdot 1 \bmod qL(\infty)$. The pair $(L(\infty), B(\infty))$ is called the *crystal base* of $U_q^-(\mathfrak{g})$. It satisfies the following properties:

(i) $L(\infty)$ is a free A-submodule of $U_q^-(\mathfrak{g})$, and $U_q^-(\mathfrak{g}) \cong \mathbf{Q}(q) \otimes_A L(\infty)$.

(ii) $B(\infty)$ is a basis of the $\mathbf{Q}$-vector space $L(\infty)/qL(\infty)$.

(iii) The endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ preserve $L(\infty)$, and so act on $L(\infty)/qL(\infty)$.

(iv) For any $i \in I$, we have $\tilde{e}_i B(\infty) \subset B(\infty) \sqcup \{0\}$ and $\tilde{f}_i B(\infty) \subset B(\infty)$.

(v) For $u, v \in B(\infty)$, we have $\tilde{f}_i u = v$ if and only if $\tilde{e}_i v = u$.

We denote by $u_\infty \in B(\infty)$ the image of 1 under the projection $L(\infty) \longrightarrow L(\infty)/qL(\infty)$, and define the weight function $wt : B(\infty) \longrightarrow P$ by $wt(b) := -(\alpha_{i_1} + \cdots + \alpha_{i_l})$ for $b = \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty$. We define integer-valued functions ε_i and φ_i on $B(\infty)$ by

$$\varepsilon_i(b) := \max \quad \{k : \tilde{e}_i^k b \neq 0\}, \ \varphi_i(b) := \langle h_i, wt(b) \rangle + \varepsilon_i(b).$$

An easy check shows that $B(\infty)$ equipped with the operators $\tilde{e}_i$ and $\tilde{f}_i$, and with the functions wt, ε_i and φ_i is a crystal.

2.4 Kashiwara embeddings of $B(\infty)$

Consider the additive group

$$\mathbf{Z}^\infty := \{(\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbf{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\}; \tag{2.25}$$

we will denote by $\mathbf{Z}^\infty_{\geq 0} \subset \mathbf{Z}^\infty$ the subsemigroup of nonnegative sequences. To the rest of this section, we fix an infinite sequence of indices $\iota = (\cdots, i_k, \cdots, i_2, i_1)$ from I such that

$$i_k \neq i_{k+1} \text{ and } \sharp\{k : i_k = i\} = \infty \text{ for any } i \in I. \tag{2.26}$$

Following Kashiwara [8], we will associate to ι a crystal structure on $\mathbf{Z}^\infty$ and the embedding of crystals

$$\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}^\infty, \tag{2.27}$$

which we call the *Kashiwara embedding*.

The crystal structure on $\mathbf{Z}^\infty$ corresponding to ι is defined as follows. Let $\vec{x} = (\cdots, x_k, \cdots, x_2, x_1) \in \mathbf{Z}^\infty$. For $k \geq 1$, we set

$$\sigma_k(\vec{x}) := x_k + \sum_{j > k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j. \tag{2.28}$$

Since $x_j = 0$ for $j \gg 0$, the form $\sigma_k(\vec{x})$ is well-defined, and $\sigma_k(\vec{x}) = 0$ for $k \gg 0$. For $i \in I$, let $\sigma^{(i)}(\vec{x}) := \max_{k:i_k=i}\sigma_k(\vec{x})$, and

$$M^{(i)} = M^{(i)}(\vec{x}) := \{k : i_k = i, \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x})\}.$$

Note that $\sigma^{(i)}(\vec{x}) \geq 0$, and that $M^{(i)} = M^{(i)}(\vec{x})$ is a finite set if and only if $\sigma^{(i)}(\vec{x}) > 0$. Now we define the maps $\tilde{e}_i : \mathbf{Z}^\infty \longrightarrow \mathbf{Z}^\infty \sqcup \{0\}$ and $\tilde{f}_i : \mathbf{Z}^\infty \longrightarrow \mathbf{Z}^\infty$ by setting

$$(\tilde{f}_i(\vec{x}))_k = x_k + \delta_{k,\min M^{(i)}}; \tag{2.29}$$

$$(\tilde{e}_i(\vec{x}))_k = x_k - \delta_{k,\max M^{(i)}} \text{ if } \sigma^{(i)}(\vec{x}) > 0; \text{ otherwise } \tilde{e}_i(\vec{x}) = 0. \qquad (2.30)$$

We also define the weight function and the functions ε_i and φ_i on $\mathbf{Z}^\infty$ by

$$wt(\vec{x}) := -\sum_{j=1}^{\infty} x_j \alpha_{i_j}, \ \varepsilon_i(\vec{x}) := \sigma^{(i)}(\vec{x}), \ \varphi_i(\vec{x}) := \langle h_i, wt(\vec{x}) \rangle + \varepsilon_i(\vec{x}).$$

An easy check shows that these maps make $\mathbf{Z}^\infty$ into a crystal. We will denote this crystal by $\mathbf{Z}_\iota^\infty$. Note that, in general, the semigroup $\mathbf{Z}_{\geq 0}^\infty$ is not a subcrystal of $\mathbf{Z}_\iota^\infty$ since it is not stable under the action of $\tilde{e}_i$'s.

The Kashiwara embedding is given by the following theorem.

Theorem 2.5 *There is a unique embedding of crystals*

$$\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}_{\geq 0}^\infty \subset \mathbf{Z}_\iota^\infty, \qquad (2.31)$$

such that $\Psi_\iota(u_\infty) = (\cdots, 0, \cdots, 0, 0)$.

Proof. The uniqueness of Ψ_ι follows from the fact that every element of $B(\infty)$ is obtained from u_∞ by a sequence of operators $\tilde{f}_i$. To prove the existence, we show that Ψ_ι can be obtained by iterating the following construction. Recall that to every $i \in I$ we associate a crystal B_i as in Example 2.4.

Theorem 2.6 ([8]) *For any $i \in I$, there is a unique embedding of crystals*

$$\Psi_i : B(\infty) \quad \hookrightarrow \quad B(\infty) \otimes B_i, \qquad (2.32)$$

such that $\Psi_i(u_\infty) = u_\infty \otimes (0)_i$.

An explicit formula for Ψ_i is given as follows. Let $x \mapsto x^*$ be the $\mathbf{Q}(q)$-algebra antiautomorphism of $U_q(\mathfrak{g})$ given by:

$$e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h}. \qquad (2.33)$$

It is proved in [8, Theorem 2.1.1] that this anti-automorphism preserves $L(\infty)$, and that the induced action on $L(\infty)/qL(\infty)$ preserves the crystal $B(\infty)$. Now for $b \in B(\infty)$ we have: $\Psi_i(b) = b' \otimes (-a)_i$, where $a = \varepsilon_i(b^*) \geq 0$ and $b' = (\tilde{e}_i^a(b^*))^*$.

Returning to the Kashiwara embedding Ψ_ι, take any $b \in B(\infty)$ and define the elements $b_0, b_1, b_2, \cdots$ of $B(\infty)$ and non-negative integers $a_1, a_2, \cdots$ recursively by:

$$b_0 = b, \ \Psi_{i_k}(b_{k-1}) = b_k \otimes (-a_k)_{i_k} \quad (k \geq 1). \qquad (2.34)$$

The definitions readily imply that $b_k = u_\infty$ and $a_k = 0$ for $k \gg 0$. Thus, the sequence $(\cdots, a_k, \cdots, a_2, a_1)$ belongs to $\mathbf{Z}_{\geq 0}^\infty$, and we set

$$\Psi_\iota(b) = (\cdots, a_k, \cdots, a_2, a_1).$$

The injectivity of Ψ_ι follows from that of the Ψ_{i_k}. To complete the proof of Theorem 2.5, it remains to show that $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}_\iota^\infty$ is an embedding of crystals. Tracing the definition of Ψ_ι and using Theorem 2.6, we only need to check the following: if we identify a sequence $(\cdots, 0, 0, a_k, \cdots, a_2, a_1) \in \mathbf{Z}_\iota^\infty$ with an element $u_\infty \otimes (-a_k)_{i_k} \otimes \cdots \otimes (-a_2)_{i_2} \otimes (-a_1)_{i_1}$ of the tensor product of crystals $B(\infty) \otimes B_{i_k} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}$ for some $k \gg 0$, then the crystal structure on $\mathbf{Z}_\iota^\infty$ agrees with that of $B(\infty) \otimes B_{i_k} \otimes \cdots \otimes B_{i_2} \otimes B_{i_1}$. This is a direct consequence of our definitions and Lemma 1.3.6 in [8]. This concludes the proof of Theorem 2.5. $\qquad\square$

Remark. The recursive definition (2.34) of the Kashiwara embedding can be reformulated as follows: if $\Psi_\iota(b) = (\cdots, a_k, \cdots, a_2, a_1)$ then each a_k is given by

$$a_k = \varepsilon_{i_k}(\tilde{e}_{i_{k-1}}^{a_{k-1}} \cdots \tilde{e}_{i_1}^{a_1} b^*) = \max \quad \{a : \tilde{e}_{i_k}^a \tilde{e}_{i_{k-1}}^{a_{k-1}} \cdots \tilde{e}_{i_1}^{a_1} b^* \neq 0\}. \qquad (2.35)$$

This is the crystal version of the *string parametrization* introduced in Section 2 of [2]. It is not hard to show that, in the terminology of [2], $\Psi_\iota(b)$ is the *string* of b^* (more precisely, of the global basis vector corresponding to b^*) in direction ι. Passing from the $\tilde{e}_i$ to the $\tilde{f}_i$ transforms (2.35) into one more equivalent description of the Kashiwara embedding: the sequence $(\cdots, a_k, \cdots, a_2, a_1) = \Psi_\iota(b)$ is uniquely determined by the conditions that

$$b^* = \tilde{f}_{i_1}^{a_1} \tilde{f}_{i_2}^{a_2} \cdots u_\infty, \text{ and } \tilde{e}_{i_{k-1}}(\tilde{f}_{i_k}^{a_k} \tilde{f}_{i_{k+1}}^{a_{k+1}} \cdots u_\infty) = 0 \text{ for } k > 1. \qquad (2.36)$$

In the rest of the paper we deal with the following

Main Problem. Describe explicitly the image of Ψ_ι.

3   Polyhedral Realizations of $B(\infty)$

3.1   Piecewise-linear transformations S_k

We will retain the notation introduced above. In particular, we fix a sequence of indices $\iota := (i_k)_{k \geq 1}$ as in (2.26). Consider the infinite dimensional vector space

$$\mathbf{Q}^\infty := \{\vec{x} = (\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbf{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$

and its dual $(\mathbf{Q}^\infty)^* := \mathrm{Hom}(\mathbf{Q}^\infty, \mathbf{Q})$. We will write a linear form $\varphi \in (\mathbf{Q}^\infty)^*$ as $\varphi(\vec{x}) = \sum_{k \geq 1} \varphi_k x_k$ $(\varphi_j \in \mathbf{Q})$.

For every $k \geq 1$, we define $k^{(+)}$ to be the minimal index j such that $j > k$ and $i_j = i_k$. We also define $k^{(-)}$ to be the maximal index j such that $j < k$ and $i_j = i_k$; if $i_j \neq i_k$ for $1 \leq j < k$, then we set $k^{(-)} = 0$. Let $\beta_k \in (\mathbf{Q}^\infty)^*$ be a linear form given by

$$\beta_k(\vec{x}) = \sigma_k(\vec{x}) - \sigma_{k^{(+)}}(\vec{x}), \qquad (3.1)$$

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where the forms σ_k are defined by (2.28). Since $\langle h_i, \alpha_i \rangle = 2$ for any $i \in I$, we have

$$\beta_k(\vec{x}) = x_k + \sum_{k < j < k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}}. \tag{3.2}$$

We will also use the convention that $\beta_0(\vec{x}) = 0$ for all $\vec{x} \in \mathbf{Q}^\infty$. Using this notation, for every $k \geq 1$, we define a piecewise-linear operator $S_k = S_{k,\iota}$ on $(\mathbf{Q}^\infty)^*$ by

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta_{k^{(-)}} & \text{if } \varphi_k \leq 0. \end{cases} \tag{3.3}$$

An easy check shows that $(S_k)^2 = S_k$.

3.2 Main theorem

For a sequence $\iota = (i_k)_{k \geq 1}$ satisfying (2.26), we denote by $\Xi_\iota \subset (\mathbf{Q}^\infty)^*$ the subset of linear forms that are obtained from the coordinate forms x_j by applying transformations $S_k = S_{k,\iota}$ (see (3.3)). In other words, we set

$$\Xi_\iota := \{ S_{j_l} \cdots S_{j_2} S_{j_1} x_{j_0} : l \geq 0, \ j_0, \cdots, j_l \geq 1 \}. \tag{3.4}$$

Recall that, for $k \geq 1$, the condition $k^{(-)} = 0$ means that $i_j \neq i_k$ for $1 \leq j < k$. We will impose on ι the following positivity assumption:

$$\text{if } k^{(-)} = 0 \text{ then } \varphi_k \geq 0 \text{ for any } \varphi = \sum \varphi_j x_j \in \Xi_\iota. \tag{3.5}$$

Now we are in a position to formulate our main result.

Theorem 3.1 *Let ι be a sequence of indices satisfying (2.26) and the positivity assumption (3.5). Let $\Psi_\iota : B(\infty) \hookrightarrow \mathbf{Z}_{\geq 0}^\infty$ be the corresponding Kashiwara embedding. Then the image $\mathrm{Im}(\Psi_\iota)$ is equal to*

$$\Sigma_\iota := \{ x \in \mathbf{Z}_{\geq 0}^\infty \subset \mathbf{Q}^\infty : \varphi(x) \geq 0 \text{ for any } \varphi \in \Xi_\iota \}. \tag{3.6}$$

Proof. In view of Theorem 2.5, the image $\mathrm{Im}(\Psi_\iota)$ is a subcrystal of $\mathbf{Z}_\iota^\infty$ obtained by applying the operators $\tilde{f}_i$ to $\Psi_\iota(u_\infty) = \vec{0} = (\cdots, 0, 0, 0)$; in particular, $\mathrm{Im}(\Psi_\iota) \subset \mathbf{Z}_{\geq 0}^\infty$. Since $\vec{0} \in \Sigma_\iota$, the inclusion $\mathrm{Im}(\Psi_\iota) \subset \Sigma_\iota$ follows from the fact that Σ_ι is closed under all $\tilde{f}_i$. Let us prove this fact. Let $\vec{x} = (\cdots, x_2, x_1) \in \Sigma_\iota$ and $i \in I$, and suppose that $\tilde{f}_i \vec{x} = (\cdots, x_k + 1, \cdots, x_2, x_1)$ (in particular, $i_k = i$). We need to show that

$$\varphi(\tilde{f}_i \vec{x}) \geq 0 \tag{3.7}$$

for any $\varphi = \sum \varphi_j x_j \in \Xi_\iota$. Since $\varphi(\tilde{f}_i \vec{x}) = \varphi(\vec{x}) + \varphi_k \geq \varphi_k$, it is enough to consider the case when $\varphi_k < 0$. By (3.5), we have $k^{(-)} \geq 1$. Remembering (2.29), we have $\sigma_k(\vec{x}) > \sigma_{k^{(-)}}(\vec{x})$ and then by (3.1), we conclude that

$$\beta_{k^{(-)}}(\vec{x}) = \sigma_{k^{(-)}}(\vec{x}) - \sigma_k(\vec{x}) \leq -1.$$

It follows that

$$\begin{aligned}
\varphi(\tilde{f}_i\vec{x}) &= \varphi(\vec{x}) + \varphi_k \\
&\geq \varphi(\vec{x}) - \varphi_k\beta_{k^{(-)}}(\vec{x}) \\
&= (S_k\varphi)(\vec{x}) \geq 0,
\end{aligned}$$

since $S_k\varphi \in \Xi_\iota$. This proves the inclusion $\mathrm{Im}\,(\Psi_\iota) \subset \Sigma_\iota$.

To prove the reverse inclusion $\Sigma_\iota \subset \mathrm{Im}\,(\Psi_\iota)$, we first show that Σ_ι is a subcrystal of $\mathbf{Z}_\iota^\infty$, i.e., that $\tilde{e}_i\Sigma_\iota \subset \Sigma_\iota \sqcup \{0\}$ for any $i \in I$. Let $\vec{x} = (\cdots, x_2, x_1) \in \Sigma_\iota$ and $i \in I$, and suppose that $\tilde{e}_i\vec{x} = (\cdots, x_k - 1, \cdots, x_2, x_1)$; in particular, $i_k = i$. We need to show that

$$\varphi(\tilde{e}_i\vec{x}) \geq 0, \tag{3.8}$$

for any $\varphi = \sum \varphi_j x_j \in \Xi_\iota$. Since $\varphi(\tilde{e}_i\vec{x}) = \varphi(\vec{x}) - \varphi_k \geq -\varphi_k$, it is enough to consider the case when $\varphi_k > 0$. Remembering (2.30), we have $\sigma_k(\vec{x}) > \sigma_{k^{(+)}}(\vec{x})$ and then by (3.1), we conclude that

$$\beta_k(\vec{x}) = \sigma_k(\vec{x}) - \sigma_{k^{(+)}}(\vec{x}) \geq 1.$$

It follows that

$$\begin{aligned}
\varphi(\tilde{e}_i\vec{x}) &= \varphi(\vec{x}) - \varphi_k \\
&\geq \varphi(\vec{x}) - \varphi_k\beta_k(\vec{x}) \\
&= (S_k\varphi)(\vec{x}) \geq 0,
\end{aligned}$$

since $S_k\varphi \in \Xi_\iota$.

To complete the proof of the inclusion $\Sigma_\iota \subset \mathrm{Im}\,(\Psi_\iota)$, we make the following observation: if $\vec{x} \in \mathbf{Z}_{\geq 0,\iota}^\infty$ and $\vec{x} \neq \vec{0}$ then $\tilde{e}_i\vec{x} \neq 0$ for some $i \in I$. (Indeed, one can take $i = i_j$, where j is the maximal index such that $x_j > 0$.). Since $\Sigma_\iota \subset \mathbf{Z}_{\geq 0}^\infty$, we conclude that every $\vec{x} \in \Sigma_\iota$ can be transformed to $\vec{0}$ by a sequence of operators $\tilde{e}_i$. By (2.11), $\vec{x}$ is obtained from $\vec{0}$ by a sequence of operators $\tilde{f}_i$, hence belongs to $\mathrm{Im}\,(\Psi_\iota)$, and we are done. $\qquad\square$

3.3 Remarks on the positivity assumption

In this subsection, we shall give some equivalent reformulations of the positivity assumption (3.5). This will allow us to sharpen Theorem 3.1. We retain all the previous notation; in particular, we fix a sequence ι satisfying (2.26).

For every $k \geq 1$, we introduce the transformations $E_k, F_k : \mathbf{Z}^\infty \longrightarrow \mathbf{Z}^\infty \sqcup \{0\}$ that act on $\vec{x} = (\cdots, x_2, x_1)$ by

$$E_k(\vec{x}) \quad := \quad \begin{cases} (\cdots, x_k - 1, \cdots, x_2, x_1) & \text{if } \beta_k(\vec{x}) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

$$F_k(\vec{x}) \quad := \quad \begin{cases} (\cdots, x_k + 1, \cdots, x_2, x_1) & \text{if } \beta_{k^{(-)}}(\vec{x}) < 0 \text{ or } k^{(-)} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Comparing these definitions with (2.29) and (2.30), we see that the operators $\tilde{e}_i$ and $\tilde{f}_i$ can be written as

$$\tilde{e}_i = E_{\max M^{(i)}}, \ \tilde{f}_i = F_{\min M^{(i)}}. \tag{3.11}$$

Let $\Phi = \Phi_\iota \subset \mathbf{Z}^\infty$ be the set of all $\vec{x}$ obtained from $\vec{0} = (\cdots, 0, 0)$ by applying transformations E_k and F_k. Let $\Phi^+ \subset \Phi$ be the set of all $\vec{x}$ obtained from $\vec{0}$ by applying the F_k. Recall also the definitions (3.4) of the set of linear forms $\Xi = \Xi_\iota$, and (3.6) of the subset $\Sigma = \Sigma_\iota \subset \mathbf{Z}_{\geq 0}^\infty$. We will say that a transformation S_k acts positively on a linear form φ if $\varphi_k > 0$, i.e., the first possibility in (3.3) is realized. We define $\Xi^+ \subset \Xi$ to be the set of forms obtained from the coordinate forms x_j by applying positive actions of the transformations S_k. Let

$$\Sigma^+ = \{x \in \mathbf{Z}_{\geq 0}^\infty : \varphi(x) \geq 0 \text{ for any } \varphi \in \Xi^+\}. \tag{3.12}$$

The sets Φ, Φ^+, Σ, and Σ^+ are related to each other and to the image $\mathrm{Im}\,(\Psi_\iota)$ of the Kashiwara embedding as follows (note that we do not assume (3.5) here).

Proposition 3.2 *We have*

$$\Sigma \subset \Sigma^+ \subset \mathrm{Im}\,(\Psi_\iota) \subset \Phi^+ \subset \Phi. \tag{3.13}$$

Proof. The inclusions $\Sigma \subset \Sigma^+$ and $\Phi^+ \subset \Phi$ are obvious. Since every $\vec{x} \in \mathrm{Im}\,(\Psi_\iota)$ is obtained from $\vec{0}$ by applying the $\tilde{f}_i$, the inclusion $\mathrm{Im}\,(\Psi_\iota) \subset \Phi^+$ follows from the second equality in (3.11). The remaining inclusion $\Sigma^+ \subset \mathrm{Im}\,(\Psi_\iota)$ is proved by the same argument as the inclusion $\Sigma_\iota \subset \mathrm{Im}\,(\Psi_\iota)$ in Theorem 3.1 (it is seen by inspection that this argument does not use (3.5)). $\square$

Remark. Note that the same argument as in the proof of Theorem 3.1 establishes the following inclusions:

$$E_k\Sigma \subset \Sigma \sqcup \{0\}, \ E_k\Sigma^+ \subset \Sigma^+ \sqcup \{0\} \text{ for any } k \geq 1. \tag{3.14}$$

Theorem 3.3 *The following conditions are equivalent:*

(i) All the inclusions in (3.13) are equalities.

(ii) $\Phi \subset \mathbf{Z}_{\geq 0}^\infty$.

(iii) Ξ *satisfies the positivity assumption (3.5).*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious since $\Phi^+ \subset \mathbf{Z}_{\geq 0}^\infty$. To prove (iii) $\Rightarrow$ (i), we notice that (3.5) implies the following companion of (3.14):

$$F_k\Sigma \subset \Sigma \sqcup \{0\} \text{ for any } k \geq 1; \tag{3.15}$$

this is proved by the same argument as the fact that Σ is closed under all $\tilde{f}_i$ in the proof of Theorem 3.1. Combining (3.14) and (3.15), we conclude that $\Phi \subset \Sigma$, which implies (i).

It remains to prove (ii) $\Rightarrow$ (iii). We will deduce this from the following lemma.

Lemma 3.4 *The set of linear forms that take nonnegative values on Φ, is closed under all transformations S_k.*

Proof. Suppose $\varphi(\vec{x}) = \sum_{k \geq 1} \varphi_k x_k \geq 0$ for all $\vec{x} \in \Phi$. Take any $\vec{x} \in \Phi$. We need to show that $S_k \varphi(\vec{x}) \geq 0$ for any $k \geq 1$. First suppose that $\varphi_k > 0$, i.e., $S_k \varphi = \varphi - \varphi_k \beta_k$. If $\beta_k(\vec{x}) \leq 0$ then $S_k \varphi(\vec{x}) \geq \varphi(\vec{x}) \geq 0$; so we can assume that $\beta_k(\vec{x}) = l > 0$. Using (3.2) and (3.9), we conclude that $(E_k)^l \vec{x} \in \Phi$, and

$$S_k \varphi(\vec{x}) = \varphi(\vec{x}) - l\varphi_k = \varphi((E_k)^l \vec{x}) \geq 0,$$

as required.

It remains to consider the case when $\varphi_k < 0$, i.e., $S_k \varphi = \varphi - \varphi_k \beta_{k(-)}$. If $\beta_k(\vec{x}) \geq 0$ then $S_k \varphi(\vec{x}) \geq \varphi(\vec{x}) \geq 0$; so we can assume that $\beta_k(\vec{x}) = -l < 0$. Using (3.2) and (3.10), we conclude that $(F_k)^l \vec{x} \in \Phi$, and

$$S_k \varphi(\vec{x}) = \varphi(\vec{x}) + l\varphi_k = \varphi((F_k)^l \vec{x}) \geq 0,$$

as required. □

Now we can complete the proof of (ii) $\Rightarrow$ (iii). By (ii) and Lemma 3.4, every form $\varphi \in \Xi$ takes nonnegative values on Φ since φ is obtained from some coordinate form x_j by applying the transformations S_k, and x_j is nonnegative on Φ. In particular, if $k^{(-)} = 0$ for some $k \geq 1$ then

$$\varphi_k = \varphi(F_k \vec{0}) \geq 0,$$

which proves (3.5). Theorem 3.3 is proved. □

Remarks. (a) Using Theorem 3.3, one can produce several other equivalent reformulations of (3.5). For instance, each of the following two conditions is also equivalent to (3.5):

(iv) $\Sigma = \Sigma^+$;

(v) $\Phi = \Phi^+$

(the implications (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (v) $\Rightarrow$ (ii) are obvious).

(b) It would be interesting to know if (3.5) holds for any symmetrizable Kac-Moody algebra and any sequence ι. This will be true in all the examples considered in the rest of the paper. In fact, in all these examples we will have $\Xi = \Xi^+$, which is stronger than the condition (iv) above.

3.4 Periodic case

In the subsequent sections we shall only treat the following special infinite sequence ι. We fix some linear ordering of the index set I, i.e., identify I with $\{1, 2, \cdots, n\}$. Then we take

$$\iota = (\cdots, \underbrace{n, \cdots, 2, 1}, \cdots, \underbrace{n, \cdots, 2, 1}, \underbrace{n, \cdots, 2, 1}).$$

In other words, $i_k = \overline{k}$, where $\overline{k} \in \{1, 2, \cdots, n\}$ is congruent to k modulo n. We call this sequence ι *periodic*. Relative to the periodic sequence, the above notation simplifies as follows.

First of all, for any $k \geq 1$ we have $k^{(+)} = k + n$; we also have $k^{(-)} = k - n$ if $k > n$, and $k^{(-)} = 0$ if $k \leq n$. The forms β_k take the form

$$\beta_k(\vec{x}) = x_k + \sum_{j=k+1}^{k+n-1} \langle h_{\overline{k}}, \alpha_{\overline{j}} \rangle x_j + x_{k+n},$$

and the transformations S_k can be written as

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k & \text{if } \varphi_k \geq 0, \\ \varphi - \varphi_k \beta_{k-n} & \text{if } k > n, \ \varphi_k < 0, \\ \varphi & \text{if } 1 \leq k \leq n, \ \varphi_k < 0. \end{cases} \tag{3.16}$$

Finally, the positivity assumption (3.5) in Theorem 3.1 takes the form

$$\varphi_i \geq 0 \text{ for any } i = 1, 2, \cdots, n \text{ and } \varphi = \sum \varphi_j x_j \in \Xi_\iota. \tag{3.17}$$

This means that, for $\varphi \in \Xi_\iota$, the third opportunity in (3.16) is never realized.

4   Rank 2 case

In this section, we specialize Theorem 3.1 to the Kac-Moody algebras of rank 2. We will give an explicit description of the image of the Kashiwara embedding. This description sharpens the one given by Kashiwara in [8, Sect.2].

Without loss of generality, we can and will assume that $I = \{1, 2\}$, and $\iota = (\cdots, 2, 1, 2, 1)$. Let the Cartan data be given by:

$$\langle h_1, \alpha_1 \rangle = \langle h_2, \alpha_2 \rangle = 2, \ \langle h_1, \alpha_2 \rangle = -c_1, \ \langle h_2, \alpha_1 \rangle = -c_2.$$

Here we either have $c_1 = c_2 = 0$, or both c_1 and c_2 are positive integers. We set $\lambda = c_1 c_2 - 2$, and define the integer sequence $a_l = a_l(c_1, c_2)$ for $l \geq 0$ by setting $a_0 = 0$, $a_1 = 1$ and, for $k \geq 1$,

$$a_{2k} = c_1 P_{k-1}(\lambda), \ a_{2k+1} = P_k(\lambda) + P_{k-1}(\lambda), \tag{4.1}$$

where the $P_k(\lambda)$ are *Chebyshev polynomials* given by

$$P_k(\alpha + \alpha^{-1}) = \frac{\alpha^{k+1} - \alpha^{-k-1}}{\alpha - \alpha^{-1}}. \tag{4.2}$$

Equivalently, the generating function for Chebyshev polynomials is given by

$$\sum_{k \geq 0} P_k(\lambda) z^k = (1 - \lambda z + z^2)^{-1}. \tag{4.3}$$

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The several first Chebyshev polynomials and terms a_l are given by

$$P_0(\lambda) = 1, \ P_1(\lambda) = \lambda, \ P_2(\lambda) = \lambda^2 - 1, \ P_3(\lambda) = \lambda^3 - 2\lambda,$$

$$a_2 = c_1, \ a_3 = c_1 c_2 - 1, \ a_4 = c_1(c_1 c_2 - 2),$$

$$a_5 = (c_1 c_2 - 1)(c_1 c_2 - 2) - 1, \ a_6 = c_1(c_1 c_2 - 1)(c_1 c_2 - 3),$$

$$a_7 = c_1 c_2 (c_1 c_2 - 2)(c_1 c_2 - 3) - 1.$$

Let $l_{\max} = l_{\max}(c_1, c_2)$ be the minimal index l such that $a_{l+1} < 0$ (if $a_l \geq 0$ for all $l \geq 0$, then we set $l_{\max} = +\infty$). By inspection, if $c_1 c_2 = 0$ (resp. $1, 2, 3$) then $l_{\max} = 2$ (resp. $3, 4, 6$). Furthermore, if $c_1 c_2 \leq 3$ then $a_{l_{\max}} = 0$ and $a_l > 0$ for $1 \leq l < l_{\max}$. On the other hand, if $c_1 c_2 \geq 4$, i.e., $\lambda \geq 2$, it is easy to see from (4.2) or (4.3) that $P_k(\lambda) > 0$ for $k \geq 0$, hence $a_l > 0$ for $l \geq 1$; in particular, in this case $l_{\max} = +\infty$.

Theorem 4.1 *In the rank 2 case, the image of the Kashiwara embedding is given by*

$$\mathrm{Im}\,(\Psi_\iota) = \left\{ (\cdots, x_2, x_1) \in \mathbf{Z}_{\geq 0}^\infty : \begin{array}{l} x_k = 0 \text{ for } k > l_{\max}, \\ a_l x_l - a_{l-1} x_{l+1} \geq 0 \text{ for } 1 \leq l < l_{\max} \end{array} \right\}. \tag{4.4}$$

Proof. We will deduce our theorem from Theorem 3.1. Thus, our first goal is to describe the set of linear forms Ξ_ι (see (3.4)), and to check the positivity assumption (3.17). For $k \geq 1$ and $0 \leq l < l_{\max}$, we set

$$\varphi_k^{(l)} = S_{k+l-1} \cdots S_{k+1} S_k x_k; \tag{4.5}$$

in particular, $\varphi_k^{(0)} = x_k$. We also define $a_l' = a_l(c_2, c_1)$, i.e., the numbers a_l' are given by (4.1) with c_1 replaced by c_2.

Lemma 4.2   (i) *If k is odd then $\varphi_k^{(l)} = a_{l+1} x_{k+l} - a_l x_{k+l+1}$; if k is even then $\varphi_k^{(l)} = a_{l+1}' x_{k+l} - a_l' x_{k+l+1}$.*

(ii) *If $c_1 c_2 \leq 3$, i.e., $l_{\max} < +\infty$, then $\varphi_k^{(l_{\max}-1)} = -x_{k+l_{\max}}$.*

(iii) *The set Ξ_ι consists of all linear forms $\varphi_k^{(l)}$ with $k \geq 1$ and $0 \leq l < l_{\max}$.*

(iv) *The set Ξ_ι satisfies the positivity assumption (3.17).*

Proof. (i) In view of periodicity, it is enough to show that $\varphi_1^{(l)} = a_{l+1} x_{l+1} - a_l x_{l+2}$ for $0 \leq l < l_{\max}$. We prove this by induction on l. The claim is obviously true for $l = 0$, since $a_0 = 0$ and $a_1 = 1$. So let us assume that the claim is true

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for some $\varphi_1^{(l)}$ such that both a_l and a_{l+1} are positive; we need to show that the claim is then true for $\varphi_1^{(l+1)}$. By (3.16),

$$\varphi_1^{(l+1)} = S_{l+1}\varphi_1^{(l)} = \varphi_1^{(l)} - a_{l+1}\beta_{l+1} = a_{l+1}x_{l+1} - a_lx_{l+2} - a_{l+1}\beta_{l+1},$$

where the forms β_l are given by

$$
\begin{aligned}
\beta_{2k-1}(\vec{x}) &= x_{2k-1} - c_1x_{2k} + x_{2k+1}, \\
\beta_{2k}(\vec{x}) &= x_{2k} - c_2x_{2k+1} + x_{2k+2}.
\end{aligned}
$$

Therefore, $\varphi_1^{(l+1)} = (c_1a_{l+1} - a_l)x_{l+2} - a_{l+1}x_{l+3}$ if l is even, and $\varphi_1^{(l+1)} = (c_2a_{l+1} - a_l)x_{l+2} - a_{l+1}x_{l+3}$ if l is odd. It remains to show that the sequence (a_l) satisfies the recursions

$$a_{2k+2} = c_1a_{2k+1} - a_{2k}, \ a_{2k+1} = c_2a_{2k} - a_{2k-1}.$$

These recursions follow from (4.1) with the help of the well-known recursion $P_k(\lambda) = \lambda P_{k-1} - P_{k-2}$ for Chebyshev polynomials (the latter recursion is an easy consequence of (4.3)).

(ii) Again it is enough to treat the case $k = 1$. By part (i), we have $\varphi_1^{(l_{\max}-1)} = a_{l_{\max}}x_{l_{\max}} - a_{l_{\max}-1}x_{l_{\max}+1}$. So we only need to show that $a_{l_{\max}} = 0$ and $a_{l_{\max}-1} = 1$ in all the cases when $c_1c_2 \leq 3$. This is just seen by inspection.

(iii) We only need to show that the set of all linear forms $\varphi_k^{(l)}$ with $k \geq 1$ and $0 \leq l < l_{\max}$ is closed under all the transformations S_j. If $l = 0$ then the only S_j that acts non-trivially on $\varphi_k^{(0)} = x_k$ is S_k, and we have $S_kx_k = \varphi_k^{(1)}$. If $1 \leq l < l_{\max}$ then, in view of (i), only S_{k+l} and S_{k+l+1} can act non-trivially on $\varphi_k^{(l)}$. By definition, $S_{k+l}\varphi_k^{(l)} = \varphi_k^{(l+1)}$. We complete the proof by showing that $S_{k+l+1}\varphi_k^{(l)} = \varphi_k^{(l-1)}$. Once again, using periodicity we can assume that $k = 1$. Using (i) and (3.16), we have

$$S_{l+2}\varphi_1^{(l)} = \varphi_1^{(l)} + a_l\beta_l = S_l\varphi_1^{(l-1)} + a_l\beta_l = (\varphi_1^{(l-1)} - a_l\beta_l) + a_l\beta_l = \varphi_1^{(l-1)},$$

as claimed.

Finally, part (iv) is an immediate consequence of (i) and (iii). □

Now we return to the proof of Theorem 4.1. Using Theorem 3.1 and parts (iii) and (iv) of Lemma 4.2, we conclude that

$$\text{Im}\,(\Psi_\iota) = \Sigma_\iota = \{(\cdots, x_2, x_1) \in \mathbf{Z}_{\geq 0}^\infty : \varphi_k^{(l)} \geq 0 \text{ for } k \geq 1, 0 \leq l < l_{\max}\}. \quad (4.6)$$

Comparing this with the desired answer (4.4), and using parts (i) and (ii) of Lemma 4.2, it only remains to show that the inequalities $\varphi_k^{(l)} \geq 0$ in (4.6) are redundant when $k > 1$ and $l < l_{\max} - 1$, that is, they are consequences of the remaining inequalities. We will prove this by showing that the inequality

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$\varphi_k^{(l-1)} \geq 0$ with $k > 1$ and $l < l_{\max}$ is a consequence of $\varphi_{k-1}^{(l)} \geq 0$. As above, by using periodicity, it suffices to show that $\varphi_1^{(l)} \geq 0$ implies $\varphi_2^{(l-1)} \geq 0$. By Lemma 4.2 (i), we have

$$\varphi_1^{(l)} = a_{l+1}x_{l+1} - a_l x_{l+2}, \quad \varphi_2^{(l-1)} = a_l' x_{l+1} - a_{l-1}' x_{l+2},$$

which easily implies that

$$a_{l+1}\varphi_2^{(l-1)} = a_l' \varphi_1^{(l)} + (a_l a_l' - a_{l+1} a_{l-1}')x_{l+2}.$$

To complete the proof, it suffices to show that $a_l a_l' - a_{l+1} a_{l-1}' > 0$ for all $l \geq 1$. In fact, we claim that the numbers a_l and a_l' satisfy the identity

$$a_l a_l' - a_{l+1} a_{l-1}' = 1 \tag{4.7}$$

for $l \geq 1$; this is a consequence of (4.1) and the following identities for Chebyshev polynomials:

$$(\lambda + 2)P_k(\lambda)^2 - (P_{k+1}(\lambda) + P_k(\lambda))(P_k(\lambda) + P_{k-1}(\lambda)) = 1,$$

$$(P_k(\lambda) + P_{k-1}(\lambda))^2 - (\lambda + 2)P_k(\lambda)P_{k-1}(\lambda) = 1$$

(the latter identities follow readily from (4.2)). Theorem 4.1 is proved. $\qquad\square$

Note that the cases when $l_{\max} < +\infty$, i.e., when the image $\mathrm{Im}\,(\Psi_\iota)$ is contained in a lattice of finite rank, are precisely those when the Kac-Moody algebra $\mathfrak{g}$ is of finite type. If $\mathfrak{g}$ is of type $A_1 \times A_1$ (resp. A_2, B_2 or C_2, G_2) then $l_{\max} = 2$ (resp. $3, 4, 6$). Not surprisingly, in each case $l_{\max}$ is the number of positive roots of $\mathfrak{g}$.

In conclusion of this section, we illustrate Theorem 4.1 by the example when $c_1 = c_2 = 2$, i.e., $\mathfrak{g}$ is the affine Kac-Moody Lie algebra of type $A_1^{(1)}$. In this case, we have $\lambda = c_1 c_2 - 2 = 2$. It follows at once from (4.3) that $P_k(2) = k+1$; hence, (4.1) gives $a_l = l$ for $l \geq 0$. We see that for type $A_1^{(1)}$, the image of the Kashiwara embedding is given by

$$\mathrm{Im}\,(\Psi_\iota) = \{(\cdots, x_2, x_1) \in \mathbf{Z}_{\geq 0}^\infty : lx_l - (l-1)x_{l+1} \geq 0 \text{ for } l \geq 1\}. \tag{4.8}$$

5 $\quad A_n$-case

In this section we shall apply Theorem 3.1 to the case when $\mathfrak{g} = \mathfrak{sl}_{n+1}$ is of type A_n. We will identify the index set I with $[1, n] := \{1, 2, \cdots, n\}$ in the standard way; thus, the Cartan matrix $(a_{i,j} = \langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is given by $a_{i,i} = 2$, $a_{i,j} = -1$ for $|i - j| = 1$, and $a_{i,j} = 0$ otherwise. We will find the image of the Kashiwara embedding $\mathrm{Im}\,(\Psi_\iota) \subset \mathbf{Z}_{\geq 0}^\infty$ for the periodic sequence

$$\iota = (\cdots, \underbrace{n, \cdots, 2, 1}, \cdots, \underbrace{n, \cdots, 2, 1}, \underbrace{n, \cdots, 2, 1}).$$

To formulate the answer, it will be convenient for us to change the indexing set for $\mathbf{Z}_{\geq 0}^{\infty}$ from $\mathbf{Z}_{\geq 1}$ to $\mathbf{Z}_{\geq 1} \times [1, n]$. We will do this with the help of the bijection $\mathbf{Z}_{\geq 1} \times [1, n] \to \mathbf{Z}_{\geq 1}$ given by $(j; i) \mapsto (j - 1)n + i$. Thus, we will write an element $\vec{x} \in \mathbf{Z}_{\geq 0}^{\infty}$ as a doubly-indexed family $(x_{j;i})_{j \geq 1, i \in [1, n]}$ of nonnegative integers. We will adopt the convention that $x_{j;i} = 0$ unless $j \geq 1$ and $i \in [1, n]$; in particular, $x_{j;0} = x_{j;n+1} = 0$ for all j.

Theorem 5.1 *In the above notation, the image* $\mathrm{Im}\,(\Psi_{\iota})$ *of the Kashiwara embedding is the set of all integer families* $(x_{j;i})$ *such that* $x_{j;i} = 0$ *for* $i + j > n + 1$, *and* $x_{1;i} \geq x_{2;i-1} \geq \cdots \geq x_{i;1} \geq 0$ *for* $1 \leq i \leq n$.

Proof. We will follow the proof of Theorem 4.1. So we first describe the set of linear forms Ξ_{ι}, and check the positivity assumption (3.17). To do this, we need some terminology and notation. For $(j; i) \in \mathbf{Z}_{\geq 1} \times [1, n]$, we will write the piecewise-linear transformation $S_{(j-1)n+i}$ as $S_{j;i}$; if $(j; i) \notin \mathbf{Z}_{\geq 1} \times [1, n]$ then $S_{j;i}$ is understood as the identity transformation. For $l \geq 0$ we set

$$S_{j;i}^{(l)} := S_{j;i+l-1} \cdots S_{j;i+1} S_{j;i}. \tag{5.1}$$

(again with the understanding that $S_{j;i}^{(0)}$ is the identity transformation). For $i \in [1, n]$, by an *i-admissible partition* we will mean an integer sequence $\lambda = (\lambda_1, \cdots, \lambda_i)$ such that $n + 1 - i \geq \lambda_1 \geq \cdots \geq \lambda_i \geq 0$ (if we represent partitions by Young diagrams in a usual way, then this condition means that the diagram of λ fits into the $i \times (n + 1 - i)$ rectangle). For every $(j; i) \in \mathbf{Z}_{\geq 1} \times [1, n]$ and an *i*-admissible partition λ, we define the linear form $\varphi_{j;i}^{(\lambda)}$ by

$$\varphi_{j;i}^{(\lambda)} = S_{j+i-1;1}^{(\lambda_i)} \cdots S_{j+1;i-1}^{(\lambda_2)} S_{j;i}^{(\lambda_1)} x_{j;i}. \tag{5.2}$$

Lemma 5.2 (i) *The forms* $\varphi_{j;i}^{(\lambda)}$ *are given by*

$$\varphi_{j;i}^{(\lambda)} = \sum_{k=1}^{i} (x_{j+k-1;i-k+1+\lambda_k} - x_{j+k;i-k+\lambda_k}). \tag{5.3}$$

(ii) *If* $\lambda_k = n + 1 - i$ *for* $k = 1, \cdots, i$ *then* $\varphi_{j;i}^{(\lambda)} = -x_{j+i;n+1-i}$.

(iii) *The set* Ξ_{ι} *consists of all linear forms* $\varphi_{j;i}^{(\lambda)}$, *where* $(j; i) \in \mathbf{Z}_{\geq 1} \times [1, n]$ *and* λ *is an i-admissible partition.*

(iv) *The set* Ξ_{ι} *satisfies the positivity assumption (3.17).*

Proof. (i) We prove (5.3) by induction on $|\lambda| = \lambda_1 + \cdots + \lambda_i$. For $|\lambda| = 0$, the sum on the right hand side of (5.3) telescopes to $x_{j;i} - x_{j+i;0} = x_{j;i}$, as

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required. So we assume that $|\lambda| > 0$. Let $k \in [1, i]$ be the maximal index such that $\lambda_k > 0$, and let $\lambda' = (\lambda_1, \cdots, \lambda_{k-1}, \lambda_k - 1, 0, \cdots, 0)$. By (5.1) and (5.2),

$$\varphi_{j;i}^{(\lambda)} = S_{j+k-1;i-k+\lambda_k} \varphi_{j;i}^{(\lambda')}.$$

By the inductive assumption, (5.3) holds for $\varphi_{j;i}^{(\lambda')}$; in particular, the coefficient of $x_{j+k-1;i-k+\lambda_k}$ in $\varphi_{j;i}^{(\lambda')}$ is equal to 1. By (3.16),

$$\varphi_{j;i}^{(\lambda)} = \varphi_{j;i}^{(\lambda')} - \beta_{j+k-1;i-k+\lambda_k}, \tag{5.4}$$

where the forms $\beta_{j;i}$ are given by

$$\beta_{j;i}(\vec{x}) = x_{j;i} - x_{j;i+1} - x_{j+1;i-1} + x_{j+1;i}. \tag{5.5}$$

Here note that $x_{j;i+1} = 0$ if $i = n$ and $x_{j+1;i-1} = 0$ if $i = 1$. Expressing the two summands on the right hand side of (5.4) via (5.3) and (5.5) respectively, we see that (5.3) is also valid for $\varphi_{j;i}^{(\lambda)}$. This completes the proof of (i).

(ii) This is just a special case of (5.3).

(iii) We only need to show that the set of forms $\varphi_{j;i}^{(\lambda)}$ is closed under all the transformations $S_{j';i'}$. An easy check using (5.3) and (5.5) shows that the action of $S_{j';i'}$ on $\varphi_{j;i}^{(\lambda)}$ can be described as follows. For an i-admissible partition $\lambda = (\lambda_1, \cdots, \lambda_i)$ and $k = 1, \cdots, i$, we denote by $\lambda \leftarrow k$ (resp. by $\lambda \rightarrow k$) the sequence obtained from λ by replacing λ_k with $\lambda_k + 1$ (resp. with $\lambda_k - 1$) provided that this sequence is an i-admissible partition; otherwise, we set $\lambda \leftarrow k = \lambda$ (resp. $\lambda \rightarrow k = \lambda$). Then we have

$$S_{j';i'}\varphi_{j;i}^{(\lambda)} = \begin{cases} \varphi_{j;i}^{(\lambda \leftarrow k)} & \text{if } (j'; i') = (j + k - 1; i - k + 1 + \lambda_k), \\ \varphi_{j;i}^{(\lambda \rightarrow k)} & \text{if } (j'; i') = (j + k; i - k + \lambda_k), \\ \varphi_{j;i}^{(\lambda)} & \text{otherwise.} \end{cases} \tag{5.6}$$

(iv) In view of (i) and (iii), it is enough to observe that the only components that can occur in $\varphi_{j;i}^{(\lambda)}$ with a negative coefficient are $x_{j+k;i-k+\lambda_k}$ for some $k \geq 1$. Since $j + k \geq 2$, (3.17) follows. This completes the proof of the lemma. $\square$

Now we can complete the proof of Theorem 5.1. Using Theorem 3.1 and parts (iii) and (iv) of Lemma 5.2, we conclude that $\text{Im}(\Psi_\iota)$ is the set of all nonnegative integer families $(x_{j;i})$ such that $\varphi_{j;i}^{(\lambda)} \geq 0$ for all $(j; i) \in \mathbf{Z}_{\geq 1} \times [1, n]$ and all i-admissible partitions λ. If $i = 1$, and $\lambda = (l)$ is a 1-admissible partition (i.e., $l \in [1, n]$) then (5.3) gives

$$\varphi_{j;1}^{(l)} = x_{j;l+1} - x_{j+1;l}. \tag{5.7}$$

Combining the inequalities $\varphi_{j;1}^{(l)} \geq 0$ with the inequalities $x_{j+i;n+1-i} \leq 0$ provided by Lemma 5.2 (ii), we obtain the desired set of inequalities in Theorem

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5.1. It remains to show that all the inequalities $\varphi_{j;i}^{(\lambda)} \geq 0$ are consequences of the ones with $i = 1$. But this follows at once from (5.3) and (5.7), which can be written as

$$\varphi_{j;i}^{(\lambda)} = \sum_{k=1}^{i} \varphi_{j+k-1;1}^{(i-k+\lambda_k)}.$$

Theorem 5.1 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6 $A_{n-1}^{(1)}$-case

In this section we shall apply Theorem 3.1 to the case when $\mathfrak{g}$ is an affine Lie algebra of type $A_{n-1}^{(1)}$ (also sometimes denoted by $\widehat{\mathfrak{sl}_n}$). We will assume that $n \geq 3$ since the case of $A_1^{(1)}$ was already treated above. We will identify the index set I with $[1, n]$ in the standard way; thus, the Cartan matrix $(a_{i,j} = \langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is given by $a_{i,i} = 2$, $a_{i,j} = -1$ for $|i - j| = 1$ or $|i - j| = n - 1$, and $a_{i,j} = 0$ otherwise. We will find the image of the Kashiwara embedding $\mathrm{Im}\,(\Psi_\iota) \subset \mathbf{Z}_{\geq 0}^\infty$ for the periodic sequence

$$\iota = (\cdots, \underbrace{n, \cdots, 2, 1}, \cdots, \underbrace{n, \cdots, 2, 1}, \underbrace{n, \cdots, 2, 1}).$$

To formulate the answer, we need some terminology and notation. For any $k \geq 1$, let $\Xi_k = \Xi_{k,\iota}$ denote the set of forms that can be obtained from x_k by a sequence of piecewise-linear transformations S_j (cf (3.4)). In dealing with Ξ_k, we will use the shorthand

$$j; i[k] := k - 1 + (j - 1)(n - 1) + i.$$

Thus, the correspondence $(j; i) \mapsto j; i[k]$ is a bijection between $\mathbf{Z}_{\geq 1} \times [1, n - 1]$ and $\mathbf{Z}_{\geq k}$. This bijection transforms the usual linear order on $\mathbf{Z}_{\geq k}$ into the *lexicographic* order on $\mathbf{Z}_{\geq 1} \times [1, n - 1]$ given by

$$(j'; i') < (j; i) \text{ if } j' < j \text{ or } j' = j,\ i' < i.$$

We will consider integer "matrices" $C = (c_{j;i})$ indexed by $\mathbf{Z}_{\geq 1} \times [1, n - 1]$, and such that $c_{j;i} = 0$ for $j \gg 0$. With every such C and any $k \geq 1$ we associate a linear form $\varphi_{C[k]}$ on $\mathbf{Z}^\infty$ given by

$$\varphi_{C[k]} = \sum_{j;i} c_{j;i} x_{j;i[k]}. \qquad\qquad\qquad (6.1)$$

For any $(j; i) \in \mathbf{Z}_{\geq 1} \times [1, n - 1]$, we set

$$s_{j;i} = s_{j;i}(C) = c_{1;i} + c_{2;i} + \cdots + c_{j;i}.$$

We will say that a matrix C (and each of the corresponding forms $\varphi_{C[k]}$) is *admissible* if it satisfies the following conditions:

$$s_{j;i} \geq 0 \text{ for } (j;i) \in \mathbf{Z}_{\geq 1} \times [1, n-1]. \tag{6.2}$$

$$s_{j;i} = \delta_{i,1} \text{ for } j \gg 0. \tag{6.3}$$

$$\sum_{(j';i') \leq (j;i)} s_{j';i'} \leq j \text{ for any } (j;i), \text{ with the equality for } j \gg 0. \tag{6.4}$$

If $s_{j;i} > 0$ then $s_{j';i'} > 0$ for some $(j';i')$ with $(j;i) < (j';i') \leq (j+1;i)$. (6.5)

As an example, fix $(j;i)$ and take C such that the only non-zero terms $s_{j';i'}$ are $s_{j;i} = j$ and $s_{j';1} = 1$ for all $j' > j$. The admissibility conditions are obviously satisfied, and the corresponding admissible forms $\varphi_{C[k]}$ are given by

$$\varphi_{C[k]} = jx_{j;i[k]} + x_{j+1;1[k]} - jx_{j+1;i[k]}.$$

In particular, if C_0 is the matrix corresponding to $(j;i) = (1;1)$ then $\varphi_{C_0[k]} = x_k$.

Theorem 6.1 *The image* $\mathrm{Im}(\Psi_\iota)$ *of the Kashiwara embedding is the set of* $\vec{x} \in \mathbf{Z}^\infty$ *such that* $\varphi_{C[k]}(\vec{x}) \geq 0$ *for all admissible forms* $\varphi_{C[k]}$.

Proof. The following lemma constitutes the main part of the proof.

Lemma 6.2 *For any* $k \geq 1$, *the set* $\Xi_k = \Xi_{k,\iota}$ *of forms that can be obtained from* x_k *by a sequence of piecewise-linear transformations* S_j, *consists of all admissible forms* $\varphi_{C[k]}$.

Before proving this lemma, let us show that it implies our theorem. In view of Theorem 3.1, it suffices to show that every admissible form satisfies the positivity assumption (3.17). In other words, we need to show that, for every admissible matrix C, all the entries $c_{1;i}$ ($1 \leq i \leq n-1$) and $c_{2;1}$ are nonnegative. By (6.2), we have $c_{1;i} = s_{1;i} \geq 0$; so it remains to show that $c_{2;1} \geq 0$. Again using (6.2), if we assume $c_{2;1} < 0$ then we must have $c_{1;1} = s_{2;1} - c_{2;1} > 0$. The proof of (3.17) is now completed by the following lemma.

Lemma 6.3 *The matrix* C_0 *with the entries* $c_{1;1} = 1$, *and* $c_{j;i} = 0$ *for* $(j;i) \neq (1;1)$, *is the only admissible matrix with* $c_{1;1} > 0$.

Proof. Combining the condition $c_{1;1} > 0$ with (6.2) and (6.4) (for $(j;i) = (1; n-1)$), we conclude that $s_{1;1} = c_{1;1} = 1$, and $s_{1;i} = 0$ for $i \neq 1$. Now (6.5) implies that $s_{2;1} > 0$. Combining the latter condition with (6.2) and (6.4) (for $(j;i) = (2; n-1)$), we conclude that $s_{2;1} = 1$, and $s_{2;i} = 0$ for $i \neq 1$. Continuing in the same manner, we conclude that $s_{j;i} = \delta_{i,1}$ for all $(j;i)$, i.e., $C = C_0$. $\square$

We now turn to the proof of Lemma 6.2. First let us show that, for any $k \geq 1$, the set of admissible forms $\varphi_{C[k]}$ is closed under all transformations $S_{j;i[k]}$. Note that the forms $\beta_{j;i[k]}$ are given by

$$\beta_{j;i[k]} = x_{j;i[k]} - x_{j;i+1[k]} - x_{j+1;i[k]} + x_{j+1;i+1[k]}, \tag{6.6}$$

with the convention that $x_{j;n[k]} = x_{j+1;1[k]}$ The definitions readily imply that $S_{j;i[k]}\varphi_{C[k]} = \varphi_{C'[k]}$, where the matrix C' is obtained from C as follows. If $c_{j;i} = 0$ then $C' = C$. Otherwise, we set $s_{j';i'} = s_{j';i'}(C)$ and $s'_{j';i'} = s_{j';i'}(C')$. Then the only values $s'_{j';i'}$ that are different from $s_{j';i'}$ are given by:

(I) If $c_{j;i} > 0$ then $s'_{j;i} = s_{j;i} - c_{j;i} = s_{j-1;i}$ and $s'_{j;i+1} = s_{j;i+1} + c_{j;i}$.

(II) If $c_{j;i} < 0$ then $s'_{j-1;i-1} = s_{j-1;i-1} - c_{j;i}$ and $s'_{j-1;i} = s_{j-1;i} + c_{j;i} = s_{j;i}$.
Note that in case (II), if $i = 1$ then (as shown above) we must have $j > 2$, and we identify the index $(j-1;i-1)$ with $(j-2;n-1)$. We need to show that in both cases, the transformation $C \mapsto C'$ preserves admissibility.

In both cases, the conditions (6.2) and (6.3) for C' are obvious. To prove that C' satisfies (6.5), we notice that, in case (I),

$$s'_{j;i+1} = s_{j;i+1} + c_{j;i} > 0, \ \ s_{j;i} = s_{j;i-1} + c_{j;i} > 0;$$

similarly, in case (II),

$$s'_{j-1;i-1} = s_{j-1;i-1} - c_{j;i} > 0, \ \ s_{j-1;i} = s_{j;i} - c_{j;i} > 0.$$

The fact that C' satisfies (6.5), is now a consequence of the following lemma.

Lemma 6.4 *Suppose the nonnegative integer families $s = (s_{j;i})$ and $s' = (s'_{j;i})$ satisfy one of the following two sets of properties:*

(i) *For some $(j;i)$, we have $s'_{j;i} = s_{j-1;i}$, $s_{j;i} > 0$, $s'_{j;i+1} > 0$, and $s'_{j';i'} = s_{j';i'}$ for $(j';i')$ different from $(j;i)$ and $(j;i+1)$.*

(ii) *For some $(j;i)$, we have $s'_{j-1;i} = s_{j;i}$, $s_{j-1;i} > 0$, $s'_{j-1;i-1} > 0$, and $s'_{j';i'} = s_{j';i'}$ for $(j';i')$ different from $(j-1;i-1)$ and $(j-1;i)$.*

Then if s satisfies (6.5), the same is true for s'.

Proof. We will prove the statement in case (i), the argument in case (ii) being very similar. The failure of (6.5) for s' means that, for some $(j_0;i_0)$, we have $s'_{j_0;i_0} > 0$ and $s'_{j';i'} = 0$ for $(j_0;i_0) < (j';i') \leq (j_0 + 1;i_0)$; we will refer to this as the $(j_0;i_0)$-violation. Since passing from s to s' only affects the entries $(j;i)$ and $(j;i+1)$, the $(j_0;i_0)$-violation for s' can only occur when $(j-1;i) \leq (j_0;i_0) \leq (j;i+1)$. Furthermore, since $s'_{j;i+1} > 0$, there is no $(j_0;i_0)$-violation for $(j-1;i) < (j_0;i_0) < (j;i+1)$. The $(j-1;i)$-violation for s' is impossible because $s'_{j;i} = s_{j-1;i} = s'_{j-1;i}$. Finally, since $s_{j;i} > 0$, the $(j;i+1)$-violation for s' would imply the $(j;i+1)$-violation or the $(j;i)$-violation for s, hence is also impossible. $\qquad\square$

Continuing the proof of Lemma 6.2, the fact that C' satisfies the remaining condition (6.4) is obvious in case (I). In case (II), the failure of (6.4) for C' can only happen when $C \mapsto C'$ affects the entries $s_{j-1;0} = s_{j-2;n-1}$ and $s_{j-1;1}$ for some $j > 2$; this possible violation of (6.4) is

$$\sum_{(j';i')<(j-1;1)} s'_{j';i'} \geq j - 1.$$

However, since C satisfies (6.4), we also have

$$\sum_{(j';i')<(j;1)} s'_{j';i'} = \sum_{(j';i')<(j;1)} s_{j';i'} \leq j - 1.$$

Combining the last two inequalities and using (6.2), we conclude that $s'_{j-1;i'} = 0$ for all $i' \in [1, n-1]$. But this contradicts the fact that C' satisfies (6.5), which we already established.

To complete the proof of Lemma 6.2, it is enough to show that every admissible matrix C' can be obtained from the matrix C_0 by a sequence of transformations of type (I) above. We introduce the linear order on the set of admissible matrices by setting $C \prec C'$ if $s_{j;i} > s'_{j;i}$ for the minimal index $(j;i)$ such that $s_{j;i} \neq s'_{j;i}$. In view of Lemma 6.3, C_0 is minimal with respect to this order. Note also that if $C \mapsto C'$ is a transformation of type (I) then $C \prec C'$.

Now let C' be an admissible matrix different from C_0. We will construct an admissible matrix C such that $C \mapsto C'$ is a transformation of type (I). By Lemma 6.3, we have $s'_{1;1} = c'_{1;1} = 0$. Let $(j_0; i) \in \mathbf{Z}_{\geq 1} \times [1, n-1]$ be the minimal index such that $s'_{j_0;i+1} > 0$ (the existence of $(j_0; i)$ is guaranteed by (6.3)). We claim that, for some $j \geq j_0$,

$$\sum_{(j;i)<(j';i')\leq(j+1;i)} s'_{j';i'} > 1. \tag{6.7}$$

Indeed, assuming that (6.7) is false for all $j \geq j_0$, we would obtain

$$\sum_{(j';i')\leq(j;i)} s'_{j';i'} \leq j - j_0 < j$$

for $j \gg 0$, which contradicts (6.4). Let $j \geq j_0$ be the minimal index satisfying (6.7). Arguing as in the proof of Lemma 6.3, we conclude that the only non-zero terms $s'_{j';i'}$ for $(j_0; i) \leq (j'; i') \leq (j; i)$ are $s'_{j';i+1} = 1$ for $j_0 \leq j' < j$. Furthermore, we have $s'_{j;i+1} > 0$, and if $s'_{j;i+1} = 1$ then $s'_{j';i'} > 0$ for some $(j'; i')$ with $(j; i+1) < (j'; i') \leq (j+1; i)$. Now we define the matrix C by setting $s_{j;i} = s_{j;i}(C) = s'_{j;i} + 1$, $s_{j;i+1} = s'_{j;i+1} - 1$, and $s_{j';i'} = s'_{j';i'}$ for $(j'; i')$ different from $(j; i)$ and from $(j; i+1)$. The definitions readily imply that C is admissible and $C \prec C'$, and that $S_{j;i} : C \mapsto C'$ is a transformation of type (I). Iterating this construction, we see that C' can be obtained from C_0 by a

sequence of transformations of type (I). This completes the proof of Lemma 6.2 and then Theorem 6.1. $\square$

Remarks. (a) A direct check using (6.6) shows that the form $\varphi_{C[k]}$ can also be written as

$$\varphi_{C[k]} = x_k - \sum_{(j;i) \geq (1;1)} d_{j;i[k]} \beta_{j;i[k]}, \tag{6.8}$$

where the coefficients $d_{j;i[k]}$ are given by

$$d_{j;i[k]} = j - \sum_{(j';i') \leq (j;i)} s_{j';i'}.$$

Thus, the meaning of (6.4) is that the sum in (6.8) is a (finite) nonnegative linear combination of the $\beta_{j;i[k]}$.

(b) It would be interesting to find the minimal set of inequalities defining $\mathrm{Im}\,(\Psi_\iota)$, i.e., to eliminate the redundant linear forms among the $\varphi_{C[k]}$.

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